

ON THE DYNAMICS AND ENTROPY OF THE PUSH-FORWARD MAP

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ABSTRACT. In this work we study the main dynamical properties of the push forward map, a transformation in the space of probabilities $\mathcal{P}(X)$ induced by a map $T : X \rightarrow X$, X a compact metric space. We also establish a connection between topological entropies of T and of the push forward map.

1. INTRODUCTION

During the last years some effort has been made in order to endow a probability space (of a given metric space) with a Riemannian manifold structure. One of the ingredients is the notion of a tangent space, that need to be defined in this case, and this motivates, for example, the work of Kloeckner. This author fix a certain metric space (the circle) and a map on this set (a dynamical system); this map induces a transformation on the probability space, known as the push forward map, and he is able to show some dynamical properties of this map as, for example, the entropy and he uses this special case in order to give a description of the tangent space of the probabilities of the circle.

Motivated by this work, we start to try to understand the relation between a dynamical systems on a compact metric space and the dynamical system induced on the probabilities: more specifically, to try to know which are the properties that are common to both transformations.

Some topological properties are inherited by the probability dynamics, but with certain losses: for example, in order to get transitivity in the probabilistic setting it is necessary to assume a very strong hypothesis, say, topological mixing for the map on the metric space.

The topological entropy, on the other hand, can be bounded below by the entropy of the map T and, if its positive, then the topological entropy on the probabilistic setting is indeed infinity.

The article is organized as follows: after giving the main concepts we present some results in a simple setting, assuming that the metric space is discrete. After this warm up we deal with some topological properties of the map Φ and, at the section 7, we address the question of topological entropy.

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2. MOTIVATION: THE DISCRETE CASE

The goal of the present work is to study the dynamics of the push forward map which arises from a continuous map $T : X \rightarrow X$, i.e., the map $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ given by $\Phi(\mu) := \mu \circ T^{-1}$. As a first case it is natural to consider the situation where X is a finite set or a discrete infinite set. In that case we see that the map T can be represented by a matrix, that we call $[T]$, and the push forward map $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is then given by the adjoint of the matrix $[T]$, i.e., $[\Phi] = [T]^*$.

2.1. The finite case. In this section we are going to consider finite spaces. We notice that in these cases X is not connected. We consider $X = \{x_1, \dots, x_n\}$, and we identify a function $f : X \rightarrow \mathbb{R}$ as a vector in \mathbb{R}^n by the linear isomorphism $\mathcal{L} : C^0(X) \rightarrow \mathbb{R}^n$ given by

$$\mathcal{L}(f) = (f(x_1), \dots, f(x_n)).$$

Then

$$C^0(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\} \cong \mathbb{R}^n,$$

and it implies that $C^0(X)' \cong (\mathbb{R}^n)^*$ whose basis will be the dual of the canonical one

$$\{\delta_{x_1}, \dots, \delta_{x_n}\},$$

i.e

$$\int f d\delta_{x_i} = f(x_i), \text{ for } i = 1, \dots, n.$$

by the identification

$$\mathcal{L}^*(\nu) = \mathcal{L}^*\left(\sum_{i=1}^n p_i \delta_{x_i}\right) = (p_1, \dots, p_n).$$

As $\mathcal{M}(X) \cong C^0(X)'$, where $\mathcal{M}(X)$ is the set of measures on X , then

$$\begin{aligned} \mathcal{P}(X) &= \left\{ \sum_{i=1}^n p_i \delta_{x_i} : 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^n p_i = 1 \right\} \\ &\cong \left\{ (p_1, \dots, p_n) \in \mathbb{R}^n : 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^n p_i = 1 \right\}. \end{aligned}$$

So, in that case, the push forward of T , i.e., the transformation Φ , is a map on the simplex $\Delta_n := \left\{ (p_1, \dots, p_n) \in \mathbb{R}^n : 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^n p_i = 1 \right\}$, $\Phi : \Delta_n \rightarrow \Delta_n$.

Given $T : X \rightarrow X$ a continuous map, we can set a $n \times n$ matrix of zero-one entries $[T]$, that represents T as follows:

$$[T] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} T(x_1) \\ \vdots \\ T(x_n) \end{pmatrix}$$

where

$$[T]_{ij} = \begin{cases} 1, & \text{if } T(x_i) = x_j \\ 0, & \text{otherwise.} \end{cases}$$

We can also identify the integrals in the original space with the inner product:

$$\int f d\nu = \langle \mathcal{L}(f), \mathcal{L}^*(\nu) \rangle = \sum_{i=1}^n p_i f(x_i).$$

In order to establish a matrix for the push forward of T we recall the formula of change of variables (see Lemma 16),

$$\int f \circ T d\nu = \int f d\Phi(\nu).$$

We claim that $\mathcal{L}(f \circ T) = [T]\mathcal{L}(f)$. Indeed, for any $1 \leq i \leq n$ we have the coordinate

$$(\mathcal{L}(f \circ T))_i = f(x_j) \text{ if } T(x_i) = x_j,$$

that is

$$(\mathcal{L}(f \circ T))_i = f(x_j) = f(T(x_i)) = ([T]\mathcal{L}(f))_i,$$

proving the equality.

From this, we have proved the following.

Proposition 1. *Let Φ be the push forward map associated to T and $[\Phi]$ his matrix as above i.e, if $\nu = \sum_{i=1}^n p_i \delta_{x_i}$ then*

$$\Phi\left(\sum_{i=1}^n p_i \delta_{x_i}\right) = \sum_{i=1}^n q_i \delta_{x_i} \Leftrightarrow [\Phi]\mathcal{L}^*(\nu) = \mathcal{L}^*(\Phi(\nu)).$$

Hence, $[\Phi] = [T]^*$ (the adjoint matrix).

Proof. We just observe that, the change of variables

$$\int f \circ T d\nu = \int f d\Phi(\nu),$$

is equivalent to

$$\langle \mathcal{L}(f \circ T), \mathcal{L}^*(\nu) \rangle = \langle \mathcal{L}(f), \mathcal{L}^*(\Phi(\nu)) \rangle.$$

We have proved that $\mathcal{L}(f \circ T) = [T]\mathcal{L}(f)$, so

$$\langle \mathcal{L}(f \circ T), \mathcal{L}^*(\nu) \rangle = \langle [T]\mathcal{L}(f), \mathcal{L}^*(\nu) \rangle = \langle \mathcal{L}(f), [T]^t \mathcal{L}^*(\nu) \rangle,$$

so

$$\langle \mathcal{L}(f), [T]^t \mathcal{L}^*(\nu) \rangle = \langle \mathcal{L}(f), \mathcal{L}^*(\Phi(\nu)) \rangle, \quad \forall f \text{ (i.e } \forall \mathbb{R}^n),$$

thus we get $[\Phi] = [T]^*$. ■

Example 2. *We consider $X = \{x_0, x_1, \dots, x_{n-1}\}$ and the map $T : X \rightarrow X$ given by*

$$T(x_i) = x_{i+1} \text{ mod } n.$$

Then the matrix of T is

$$[T] = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Given $\nu = \sum_{i=0}^{n-1} p_i \delta_{x_i} \in \mathcal{P}(X)$, if $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is the push forward of T , then

$$\Phi(\nu) = \sum_{i=0}^{n-1} p_i \delta_{x_{i+1} \text{ mod } n}.$$

If we consider ν as the vector $\nu = (p_0, \dots, p_{n-1})$, we see that $\Phi(p_0, \dots, p_{n-1}) = (p_{n-1}, p_1, p_2, \dots, p_0)$. Then we conclude that

$$[\Phi] = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 1 & \dots & 0 \\ 0 & \vdots & \ddots & \ddots & 1 & \vdots \\ 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = [T]^*$$

For the analogous of maps of degree d on S^1 we have:

Example 3. Let $X = \{x_0, x_1, x_2, x_3\}$ and $T : X \rightarrow X$, given by

$$T(x_i) = x_{2i \bmod 4}.$$

Then we have that $T(X) = \{x_0, x_2\}$. Given $\nu = \sum_{i=0}^3 p_i \delta_{x_i} \in \mathcal{P}(X)$, if $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is the push forward of T , we can see that

$$\Phi(\nu) = (p_0 + p_2) \delta_{x_0} + (p_1 + p_3) \delta_{x_2}.$$

If we consider the measure ν as the vector $[\nu] = (p_0, p_1, p_2, p_3)^t$ then

$$\Phi(\nu) = [\Phi][\nu],$$

where

$$[\Phi][\nu] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

So, we get $[\Phi]$, which is equal to the adjoint $[T]^*$.

2.2. The infinite case. In this section we will consider a set X infinite and discrete. In that case we know that $X = \{x_1, x_2, \dots\}$ is a countable set. We endow X with the discrete topology. We have that the distance on X given by

$$d(x_n, x_m) = \begin{cases} 1, & \text{if } n \neq m \\ 0, & \text{otherwise} \end{cases},$$

generates the discrete topology on X , and with this topology X is not compact. It is not difficult to see that the set of probability measures on X is given by

$$\mathcal{P}(X) = \left\{ \sum_{i=1}^{\infty} p_i \delta_{x_i} : x_i \in X, p_i \geq 0, \sum_{i=1}^{\infty} p_i = 1 \right\},$$

and it is also a non compact set.

Let us consider a map $T : X \rightarrow X$ and $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ its push forward. As in the finite case, we can associate to T a zero-one matrix, but now it is an infinite matrix. Again, if $[T]$ is the matrix associated to the map T we have that the matrix associated to Φ satisfies the condition $[\Phi] = [T]^*$.

As $\mathcal{P}(X)$ is convex but not compact, we can not apply the *Schauder Fixed Point Theorem*, but we have the following:

Theorem 4. Let $T : X \rightarrow X$ be a map and $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ its push forward. Then T has a periodic point if and only if Φ has a fixed point.

Proof. If there exists $p \in X$ and $n \in \mathbb{N}$ such that $T^n(p) = p$, then we have that

$$\mu = \frac{1}{n} (\delta_x + \delta_{T(x)} + \dots + \delta_{T^{n-1}(x)}) \in \mathcal{P}(X)$$

is a fixed point to Φ .

For the converse, we will divide in two cases, the first one where T is a bijection. It is also possible to think on T as a map from \mathbb{N} to \mathbb{N} , i.e, $T : \mathbb{N} \rightarrow \mathbb{N}$ by means of the identification $T(x_i) = x_j \leftrightarrow T(i) = j$. If $\mu = \sum_{i=1}^{\infty} p_i \delta_i$ is such that $\Phi(\mu) = \mu$, then

$$\mu = \sum_{i=1}^{\infty} p_i \delta_i = \sum_{i=1}^{\infty} p_i \delta_{T(i)} = \sum_{i=1}^{\infty} p_{T^{-1}(i)} \delta_i$$

So we have that $p_i = p_{T^{-1}(i)}$, for all $i \in \mathbb{N}$. As μ is a probability measure there exists $p_j \neq 0$. Since $\Phi(\mu) = \mu$, $\Phi^k(\mu) = \mu$ and it implies that $p_j = p_{T^{-k}(j)}$ for all $k \in \mathbb{N}$. If $T^{-k}(j) \neq j$ for all $k \in \mathbb{N}$, we have that the set $\{T^{-k}(j) = j_k : k \in \mathbb{N}\}$ is an infinite subset of \mathbb{N} , and we can write μ as the following

$$\mu = \sum_{k=1}^{\infty} p_{j_k} \delta_{j_k} + \sum_{i \neq j_k \forall k} p_i \delta_i = \sum_{k=1}^{\infty} p_{T^{-k}(j)} \delta_{j_k} + \sum_{i \neq j_k \forall k} p_i \delta_i = \sum_{k=1}^{\infty} p_j \delta_{j_k} + \sum_{i \neq j_k \forall k} p_i \delta_i,$$

and it implies that $\mu(\mathbb{N}) = \infty$, which is a contradiction.

For the second case we suppose T is a non bijective map; again we can think on T as a map from \mathbb{N} to \mathbb{N} . Let $\mu = \sum_{i=1}^{\infty} p_i \delta_i \in \mathcal{P}(X)$ the fixed point of Φ . As $\Phi^k(\mu) = \mu$ for all $k \in \mathbb{N}$, we have that

$$\mu = \sum_{i=1}^{\infty} p_i \delta_i = \Phi^k(\mu) = \sum_{i=1}^{\infty} p_i^k \delta_i,$$

where $p_i^k = \sum_{l=1}^{\infty} p_{i_l^k} = p_i$ is given by the set $T^{-k}(i) = \{i_1^k, i_2^k, i_3^k, \dots\}$. We know that there exists p_j such that $p_j \neq 0$. If $T^{-n}(j) \cap T^{-m}(j) \neq \emptyset$ with $m < n$, then there exists $i \in \mathbb{N}$ such that $T^n(i) = T^m(i)$, and it implies that $T^{n-m}(T^m(i)) = T^m(i)$, i.e, $T^m(i)$ is a periodic point for T . If $T^{-n}(j) \cap T^{-m}(j) = \emptyset$ with $m \neq n$, then we can write μ as the following

$$\mu = \sum_{j_l^1 \in T^{-1}(j)} p_{j_l^1}^1 \delta_{j_l^1} + \sum_{j_l^2 \in T^{-2}(j)} p_{j_l^2}^2 \delta_{j_l^2} + \sum_{j_l^3 \in T^{-3}(j)} p_{j_l^3}^3 \delta_{j_l^3} + \dots + \sum_{i \notin T^{-k}(j), \forall k \in \mathbb{N}} p_i \delta_i.$$

It implies that $\mu(X) = \infty$, because $\sum_{j_l^k \in T^{-k}(j)} p_{j_l^k}^k = p_j$ for all $k \in \mathbb{N}$, and it is a

contradiction. Then there exist $m, n \in \mathbb{N}$ such that $T^{-n}(j) \cap T^{-m}(j) \neq \emptyset$, and by the above, we get that T has a periodic point. \blacksquare

Example 5. Let $X = \{x_1, x_2, \dots\}$, $x_i \neq x_j$ for $i \neq j$, and $T : X \rightarrow X$ given by $T(x_i) = x_{i+1}$. Then, since T has no periodic point, by Theorem 4 we see that Φ has no fixed point.

3. THE PUSH FORWARD MAP Φ AND SOME METRICS ON $\mathcal{P}(X)$

Let X a connected compact separable metric space. If we consider a continuous map $T: X \rightarrow X$ it induces a map

$$\Phi: \mathcal{P}(X) \rightarrow \mathcal{P}(X),$$

where $\Phi(\mu)(A) = \mu(T^{-1}(A))$. This map is called *the push forward of T* . We are interested in the study of the dynamics of the map Φ . To do it we observe that there are metrics on $\mathcal{P}(X)$, whose make this set a compact metric space, since X is also compact.

Proposition 6. *If we consider $\mathcal{P}(X)$ with the weak topology and T is continuous, Φ is continuous. If T is an homeomorphism then Φ is an homeomorphism.*

Proof. See [4]. ■

We are interested in three particular metrics on $\mathcal{P}(X)$. The first one is the Prokhorov metric, defined by

$$d_P(\nu, \mu) = \inf\{\alpha > 0 : \mu(A) \leq \nu(A_\alpha) + \alpha \text{ and } \nu(A) \leq \mu(A_\alpha) + \alpha, \forall A \in \mathcal{B}(X)\},$$

where $A_\alpha := \{x \in X : d(x, A) < \alpha\}$. The second one is the *weak-* distance* (on a locally compact metric space) defined by

$$d(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_X g_i(x) d\mu - \int_X g_i(x) d\nu \right|,$$

where $g_i: X \rightarrow [0, 1]$ is continuous for all $i \in \mathbb{N}$ and $\{g_i\}_{i \in \mathbb{N}}$ is an enumerable dense set in $C(X, [0, 1])$. The last one is the *Wasserstein metric*, defined by

$$W_p(\mu, \nu) = \left(\inf_{\Pi} \left\{ \int_{X \times X} d^p(x, y) d\Pi \right\} \right)^{\frac{1}{p}},$$

where Π is a transport from μ to ν , say, a probability on $X \times X$ whose marginals are μ and ν .

Lemma 7. (i) *All the metrics above generates the weak topology,*
(ii) *If X is a compact Polish space, then $\mathcal{P}(X)$ with any of the above metrics is a compact Polish space.*

Proof. See [7]. ■

4. BASIC TOPOLOGICAL PROPERTIES OF THE MAP Φ

We start this section observing that Φ has a fixed point, since T is continuous.

Proposition 8. *If T is a continuous map, then Φ has a fixed point.*

Proof. We notice that Φ is a continuous map and $\mathcal{P}(X)$ is a compact convex set. By *Schauder fixed point theorem* we have that Φ has a fixed point. ■

Remark 9. *Proposition 8 implies that the set of probability measures on X which are T -invariant, denoted by $\mathcal{M}(T, X)$, is not empty.*

Proposition 10. *Let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ be topologically conjugated dynamical systems. Then $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and $\Psi : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ are topologically conjugated dynamical systems, where Φ is induced by T and Ψ is induced by S .*

Proof. Let $H : X \rightarrow Y$ be the conjugation between T and S . Then we have

$$H \circ T = S \circ H.$$

Consider the map $\Sigma : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, given by $\Sigma(\mu)(A) = \mu(H^{-1}(A))$. Then Σ is a homeomorphism. Take $\nu \in \mathcal{P}(Y)$ and see that

$$\begin{aligned} \Sigma \circ \Phi \circ \Sigma^{-1}(\mu) &= \Sigma \circ \Phi(\mu \circ H) = \Sigma(\mu \circ H \circ T) \\ &= \mu \circ H \circ T \circ H^{-1} = \mu \circ S \circ H \circ H^{-1} \\ &= \mu \circ S = \Psi(\mu) \end{aligned}$$

Hence

$$\Sigma \circ \Phi \circ \Sigma^{-1} = \Psi,$$

which implies the result. ■

Now we define a measurable partition of the set X that we call grid.

Lemma 11. *Given X a compact metric space and $\delta > 0$, there exists a measurable covering of X , $\{P_j\}_{j=1}^N$, such that each P_j has non-empty interior, $P_i \cap P_j = \emptyset$ for any $i \neq j$ and $d(x, y) < \delta$ for all $x, y \in P_j$, for all j . Moreover, there exist $\varepsilon > 0$ and points $p_i \in P_i$ such that $B_\varepsilon(p_i) \subset P_i$.*

Proof. Given $\delta > 0$, there exist $x_1, \dots, x_k \in X$ such that $X = \cup_{j=1}^k B_{\frac{\delta}{2}}(x_j)$. So we define

$$\begin{aligned} P_1 &= B_{\frac{\delta}{2}}(x_1), \\ P_2 &= (B_{\frac{\delta}{2}}(x_2)) - (B_{\frac{\delta}{2}}(x_1)) \\ &\vdots \\ P_k &= (B_{\frac{\delta}{2}}(x_k)) - (\cup_{j=1}^{k-1} B_{\frac{\delta}{2}}(x_j)). \end{aligned}$$

Then we get $X = \cup_{j=1}^k P_j$, and $P_i \cap P_j = \emptyset$, if $i \neq j$. As $P_j \subset B_{\frac{\delta}{2}}(x_j)$, $d(x, y) < \delta$ for all $x, y \in P_j$.

As the covering $X = \cup_{j=1}^k B_{\frac{\delta}{2}}(x_j)$ is finite and by construction of each P_i , we can take a suitable $\varepsilon > 0$ and choose points $p_i \in P_i$ such that $B_\varepsilon(p_i) \subset P_i$ for $i \in \{1, \dots, k\}$. ■

With this grid in mind we can approximate any measure as follows:

Lemma 12. *Given $\mu \in \mathcal{P}(X)$ and $\varepsilon > 0$, there exists*

$$\nu = \sum_{i=1}^N a_i \delta_{p_i}$$

such that $d(\mu, \nu) < \varepsilon$.

Proof. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{i=n_0+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}.$$

Using the continuity of g_i , there exists $\delta = \delta\left(n_0, \frac{\varepsilon}{2}\right)$, such that

$$d(x, y) < \delta \Rightarrow |g_i(x) - g_i(y)| < \frac{\varepsilon}{2}, \quad \forall i \in \{1, \dots, n_0\}.$$

Given $\delta > 0$, let us consider a grading $P = \{P_1, \dots, P_N\}$ such that $\text{diam}(P_i) < \delta$ for all P_i . Take points $p_i \in P_i$ and consider the probability

$$\nu = \sum_{i=1}^N \mu(P_i) \delta_{p_i}.$$

Then we notice that

$$\begin{aligned} d(\nu, \mu) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_X g_j(x) d\nu - \int_X g_j(x) d\mu \right| \\ &= \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \sum_{i=1}^k \int_{P_i} g_j(p_i) - g_j(x) d\mu \right| \\ &\leq \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=1}^k \int_{P_i} |g_j(p_i) - g_j(x)| d\mu < \varepsilon \end{aligned}$$

■

For the next we assume that the homeomorphism T is such that its periodic points are dense in X , i.e.: given $\delta > 0$, there exists a K -periodic point $p \in X$ such that its orbit $\{p, T(p), \dots, T^{K-1}(p)\}$ is δ -dense. We can also define periodic measures, say, measures that are periodic points of the dynamics Φ .

Proposition 13. *If $T : X \rightarrow X$ is a homeomorphism with dense periodic points, then the periodic points for Φ are dense in $\mathcal{P}(X)$.*

Proof. Given any measure $\mu \in \mathcal{P}(X)$, we need to show how it can be approximated by a periodic measure. Take $\varepsilon > 0$, then there exists n_0 such that

$$\sum_{i=n_0+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}.$$

Using the continuity of g_i , there exists $\delta = \delta\left(n_0, \frac{\varepsilon}{2}\right)$, such that

$$d(x, y) < \delta \Rightarrow |g_i(x) - g_i(y)| < \frac{\varepsilon}{2}, \quad \forall i \in \{1, \dots, n_0\}.$$

We consider a δ -grid on X , $P = \{P_1, \dots, P_K\}$, and take a periodic orbit in X , $\{p, T(p), \dots, T^{N-1}(p)\}$, which is $\frac{\delta}{2}$ -dense. Clearly, there exists at least one point of the orbit in each element P_i (and so $K \leq N$). Let us relabel the orbit as follows:

call q_1 a point lying in P_1 (any one of the finite points in this set can be chosen); q_i some point lying in P_i and so on, until $q_N \in P_N$. So we define the measure

$$\mu' = \sum_{i=1}^N \mu(P_i) \delta_{q_i}.$$

Then we have that

$$\begin{aligned} d(\mu, \mu') &= \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_X g_i(x) d\mu - \int_X g_i(x) d\mu' \right| \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \sum_{j=1}^K \int_{P_j} (g_i(x) - g_i(q_j)) d\mu \right| \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{j=1}^K \int_{P_j} |g_i(x) - g_i(q_j)| d\mu \\ &= \sum_{i=1}^{n_0} \frac{1}{2^i} \sum_{j=1}^K \int_{P_j} |g_i(x) - g_i(q_j)| d\mu + \sum_{i=n_0+1}^{\infty} \frac{1}{2^i} \sum_{j=1}^K \int_{P_j} |g_i(x) - g_i(q_j)| d\mu \\ &\leq \sum_{i=1}^{n_0} \frac{1}{2^i} \sum_{j=1}^K \frac{\varepsilon}{2} \mu(P_j) + \sum_{i=n_0+1}^{\infty} \frac{1}{2^i} \sum_{j=1}^K 2\mu(P_j) < \varepsilon, \end{aligned}$$

where the last inequality comes from the fact $\mu(X) = \sum_{j=1}^K \mu(P_j) = 1$ ■

Definition 14. Let $T : X \rightarrow X$ a homeomorphism of a compact metric space. We say that T is equicontinuous if the sequence of iterates of T , $\{T^n\}_{n \in \mathbb{N}}$, is an equicontinuous sequence of homeomorphisms.

Proposition 15. If T is equicontinuous, then Φ is equicontinuous.

Proof. Let us suppose T equicontinuous and consider the sequence $\{\Phi^n\}_{n \in \mathbb{N}}$. Take $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(T^n(x), T^n(y)) < \varepsilon.$$

By considering the Prokhorov metric we have

$$d_P(\mu, \nu) < \delta \Rightarrow d_P(\Phi^n(\mu), \Phi^n(\nu)) < \varepsilon, \forall n \in \mathbb{N}.$$

To see that we suppose $d_P(\mu, \nu) < \delta$ and observe that

$$(T^{-n}(A))_{\delta} \subset T^{-n}(A_{\varepsilon}),$$

where $A_{\gamma} = \{x \in X : d(x, A) < \gamma\}$, for some $A \subset X$. In fact, if $x \in (T^{-n}(A))_{\delta}$, then there exists $z \in T^{-n}(A)$ such that $d(x, z) < \delta$, but it implies $d(T^n(x), T^n(z)) < \varepsilon$. As $z \in T^{-n}(A)$, $T^n(z) \in A$, so $T^n(x) \in A_{\varepsilon}$ and it implies $x \in T^{-n}(A_{\varepsilon})$. Then we have that

$$\Phi^n(\mu)(A) = \mu(T^{-n}(A)) \leq \nu((T^{-n}(A))_{\delta}) + \delta \leq \nu(T^{-n}(A_{\varepsilon})) + \varepsilon = \Phi^n(\nu)(A_{\varepsilon}) + \varepsilon$$

$$\Phi^n(\nu)(A) = \nu(T^{-n}(A)) \leq \mu((T^{-n}(A))_{\delta}) + \delta \leq \mu(T^{-n}(A_{\varepsilon})) + \varepsilon = \Phi^n(\mu)(A_{\varepsilon}) + \varepsilon,$$

and it implies $d_P(\Phi^n(\mu), \Phi^n(\nu)) < \varepsilon$. ■

We also can prove that T Lipschitz implies Φ Lipschitz. In order to prove that result we need the following:

Lemma 16. (*Change of variables*) Let $f : X \rightarrow \mathbb{R}$ be a measurable function and $T : X \rightarrow X$ continuous. Then

$$\int_X f d(\Phi(\mu)) = \int_X (f \circ T)(x) d\mu.$$

Proof. See [7]. ■

Proposition 17. If $T : X \rightarrow X$ is C -Lipschitz, then $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is C -Lipschitz with respect to the Wasserstein metric. If we consider the Prokhorov metric or the weak-* metric Φ is Lipschitz, but C can change.

Proof. Let us consider the map $(T, T) : X \times X \rightarrow X \times X$ defined by $(T, T)(x, y) = (T(x), T(y))$. We have that (T, T) is continuous, so (T, T) induces a continuous map on $\mathcal{P}(X \times X)$, let us say Ψ . Hence if Π is a measure on $X \times X$ we have, by the Lemma 16

$$\int_{X \times X} d^p(x, y) d(\Psi(\Pi)) = \int_{X \times X} d^p(T(x), T(y)) d\Pi.$$

We observe that if $\mu, \nu \in \mathcal{P}(X)$ and Π is a transport from μ to ν then $\Psi(\Pi)$ is a transport from $\Phi(\mu)$ to $\Phi(\nu)$. Then, if T is a C -Lipschitz function we have

$$\begin{aligned} W_p^p(\Phi(\mu), \Phi(\nu)) &= \inf_{\Pi'} \left\{ \int_{X \times X} d^p(x, y) d\Pi' : \Pi' \text{ is a transport from } \Phi(\mu) \text{ to } \Phi(\nu) \right\} \\ &\leq \inf_{\Pi} \left\{ \int_{X \times X} d^p(x, y) d(\Psi(\Pi)) : \Pi \text{ is a transport from } \mu \text{ to } \nu \right\} \\ &= \inf_{\Pi} \left\{ \int_{X \times X} d^p(T(x), T(y)) d(\Pi) : \Pi \text{ is a transport from } \mu \text{ to } \nu \right\} \\ &\leq C \inf_{\Pi} \left\{ \int_{X \times X} d^p(x, y) d(\Pi) : \Pi \text{ is a transport from } \mu \text{ to } \nu \right\} \\ &= CW_p^p(\mu, \nu). \end{aligned}$$

Since the Prokhorov metric and the weak-* are equivalents to the Wasserstein metric, we get the result. ■

Another natural question is whether transitivity of T implies transitivity of Φ . The example below shows that the answer is *negative*.

Remark 18. T transitive does not imply Φ transitive.

Proof. If $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the irrational rotation on the circle given by $T(x) = x + \alpha$, α an irrational number, we have that T is transitive. As T is a translation, we have that Φ is 1-Lipschitz, if we consider on $\mathcal{P}(X)$ the Prokhorov distance. If we assume Φ transitive we have that there exists $\mu \in \mathcal{P}(X)$ such that the forward orbit $\{\Phi^n(\mu) : n \in \mathbb{N}\}$ is dense in $\mathcal{P}(X)$. Take $\varepsilon > 0$ such that $0 \notin A = \left(\varepsilon, 1 - \varepsilon\right)$ and $1 - 2\varepsilon > \varepsilon$ (what corresponds to a choice of $\varepsilon \in (0, 1/3)$). Consider the Lebesgue measure $\lambda \in \mathcal{P}(X)$, there exists $n \in \mathbb{N}$, such that $d_P(\Phi^n(\mu), \lambda) < \frac{\varepsilon}{4}$. Take

$p = 0 \in \mathbb{S}^1$. By the density of the sequence $\{\Phi^k(\mu)\}_{k \in \mathbb{N}}$, there exists $l \in \mathbb{N}$, such that $d_P(\Phi^{n+l}(\mu), \delta_0) < \frac{\varepsilon}{4}$. As Φ is 1-Lipschitz and λ is Φ -invariant, we have that

$$d_P(\Phi^{n+l}(\mu), \lambda) = d_P(\Phi^{n+l}(\mu), \Phi^l(\lambda)) \leq d_P(\Phi^n(\mu), \lambda) < \frac{\varepsilon}{4}.$$

By triangular inequality we get the following

$$d_P(\lambda, \delta_0) \leq d_P(\Phi^{n+l}(\mu), \lambda) + d_P(\Phi^{n+l}(\mu), \delta_0) \leq \frac{\varepsilon}{2}.$$

It implies that

$$\lambda(A) \leq \delta_0(A_{\frac{\varepsilon}{2}}) + \frac{\varepsilon}{2}, \text{ and } \delta_0(A) \leq \lambda(A_{\frac{\varepsilon}{2}}) + \frac{\varepsilon}{2}, \forall A \in \mathcal{B}(\mathbb{S}^1).$$

In particular, if $A = (\varepsilon, 1 - \varepsilon)$, $0 \notin A_{\frac{\varepsilon}{2}}$. Then

$$1 - 2\varepsilon = \lambda(A) \leq \delta_0(A_{\frac{\varepsilon}{2}}) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

which is a contradiction. ■

We assume now an stronger condition, say, that T is topologically mixing, i.e., given U, V open sets in X , there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n > N$. We notice that T^{-1} is also topologically mixing, since T is bijective.

Proposition 19. *If $T : X \rightarrow X$ is topologically mixing then Φ is topologically mixing.*

Proof. We notice that given $k \in \mathbb{N}$ we have that the map

$$T^k := (T, \dots, T) : X^k \rightarrow X^k$$

is topologically mixing if and only if T is topologically mixing. The proof is left to the reader.

If we take $\mu, \nu \in \mathcal{P}(X)$ and $\varepsilon > 0$ and consider the open balls $B(\mu, \varepsilon)$ and $B(\nu, \varepsilon)$ in $\mathcal{P}(X)$, then there exist $\mu' = \sum_{i=1}^k a_i \delta_{x_i} \in B(\mu, \varepsilon)$ and $\nu' = \sum_{i=1}^k b_i \delta_{y_i} \in B(\nu, \varepsilon)$. Taking the points $(x_1, \dots, x_k), (y_1, \dots, y_k) \in X^k$ and $\delta > 0$ such that

$$d((u_1, \dots, u_k), (v_1, \dots, v_k)) < \delta \Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=1}^k |f_j(u_i) - f_j(v_i)| \leq \varepsilon_0,$$

where ε_0 is such that $d(\mu, \mu') + \varepsilon_0 \leq \varepsilon$ and $d(\nu, \nu') + \varepsilon_0 \leq \varepsilon$.

Now we consider the open balls $B((x_1, \dots, x_k), \delta)$ and $B((y_1, \dots, y_k), \delta)$ in X^k . As T^k is topologically mixing there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow (T^k)^n(B((x_1, \dots, x_k), \delta)) \cap B((y_1, \dots, y_k), \delta) \neq \emptyset.$$

Then there exists $(z_1, \dots, z_k) \in B((x_1, \dots, x_k), \delta)$, such that $(T^k)^n(z_1, \dots, z_k)$ is in $B((y_1, \dots, y_k), \delta)$. Finally we consider the measure $\bar{\mu} = \sum_{i=1}^k a_i \delta_{z_i}$. As

$$d((x_1, \dots, x_k), (z_1, \dots, z_k)) < \delta \Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=1}^k |f_j(x_i) - f_j(z_i)| \leq \varepsilon_0,$$

and

$$d((T^n(z_1), \dots, T^n(z_k)), (y_1, \dots, y_k)) < \delta \Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=1}^k |f_j(T^n(z_i)) - f_j(y_i)| \leq \varepsilon_0,$$

we get

$$d(\bar{\mu}, \mu) \leq \varepsilon, \text{ and } d(\nu, \Phi^n(\bar{\mu})) \leq \varepsilon.$$

It implies that $\Phi^n(B(\mu, \varepsilon)) \cap B(\nu, \varepsilon) \neq \emptyset$. ■

Remark 20. *Is is well known that any topologically mixing continuous transformation on a compact set is transitive; then we conclude that T topologically mixing implies that Φ is transitive.*

5. LIMIT SETS

In this section we consider some limit sets for the map T and the consequences on the induced push-forward map.

5.1. Non-wandering set.

Definition 21. *Given $p \in X$, p is called non-wandering if for all U neighborhood of p and $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n > N$ and $T^n(U) \cap U \neq \emptyset$.*

Proposition 22. *If $p \in X$ is non-wandering, then δ_p is non-wandering.*

Proof. Let p be non-wandering. Then given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $T^n(B_\varepsilon(p)) \cap B_\varepsilon(p)$, i.e., there exists $q \in T^n(B_\varepsilon(p)) \cap B_\varepsilon(p)$. Now we take δ_q and notice that

$$d_P(\delta_p, \delta_q) \leq d(p, q) \Rightarrow \delta_q \in B_\varepsilon(\delta_p),$$

and as $q \in T^n(B_\varepsilon(p))$, there exists $x \in B_\varepsilon(p)$, such that $q = T^n(x)$. Then

$$\delta_q = \delta_{T^n(x)} = \Phi^n(\delta_x) \in \Phi^n(B_\varepsilon(\delta_p)).$$

Finally we conclude that $\delta_q \in B_\varepsilon(\delta_p) \cap \Phi^n(B_\varepsilon(\delta_p)) \neq \emptyset$. ■

5.2. ω -limit.

Definition 23. *Let $T : X \rightarrow X$ a continuous map. Let $x \in X$. A point $y \in X$ is an ω -limit point if there exists a sequence of natural numbers $n_k \rightarrow \infty$ (as $k \rightarrow \infty$) such that $T^{n_k}(x) \rightarrow y$. The ω -limit set is the set $\omega(x)$ of all ω -limit points.*

Proposition 24. *If $q \in \omega(p)$, then $\delta_q \in \omega(\delta_p)$.*

Proof. We need to show that there exists a sequence $\{\Phi^{n_k}(\delta_p)\}_{n_k \in \mathbb{N}}$, such that, $n_k \rightarrow \infty$ and $\Phi^{n_k}(\delta_p) \rightarrow \delta_q$. Since $q \in \omega(p)$, there exists a sequence $\{T^{n_k}(p)\}_{n_k \in \mathbb{N}}$, such that, $T^{n_k}(p) \rightarrow q$. Now given $g \in C(X)$ we have that

$$\left| \int_X g(x) d(\Phi^{n_k}(\delta_p)) - \int_X g(x) d(\delta_q) \right| = |g(T^{n_k}(p)) - g(q)|.$$

As g is continuous and $T^{n_k}(p) \rightarrow q$, $g(T^{n_k}(p)) \rightarrow g(q)$. Then we get

$$\int_X g(x) d(\Phi^{n_k}(\delta_p)) \rightarrow \int_X g(x) d(\delta_q), \quad \forall g \in C(X).$$

Hence

$$d(\Phi^{n_k}(\delta_p), \delta_q) \rightarrow 0. \quad \blacksquare$$

Definition 25. A point $p \in X$ is called recurrent if $x \in \omega(x)$. The set $\mathcal{R}(T)$ of recurrent points is T -invariant.

Hence, by Proposition 24, given $x \in \mathcal{R}(T)$, we have that $\delta_x \in \omega(\delta_x)$. Then

$$x \in \mathcal{R}(T) \Rightarrow \delta_x \in \mathcal{R}(\Phi).$$

6. ATTRACTORS

Here we are interested in know what happens with the dynamics Φ when the dynamics T has an attractor. We divide our study in two cases: the first one consists in a map T that has a point p as an attractor and the second one consists in a map that has a uniform attractor.

6.1. Point attractor.

Lemma 26. Let $T : X \rightarrow X$ be a continuous map such that $T : X \rightarrow T(X)$ is a homeomorphism. If $\lim_{n \rightarrow \infty} T^n(x) = p$, for all $x \in X$, then the sequence of maps $\{T^n\}_{n \in \mathbb{N}}$ converges uniformly to the constant map $F : X \rightarrow X$, $F(x) = p$ for all $x \in X$.

Proof. Consider the following sequence of continuous maps $G_n = T^n : X \rightarrow X$ and the map $F : X \rightarrow X$ given by $F(x) = p$ for all $x \in X$. We observe that $G_n(x) \rightarrow p$ for all $x \in X$, i.e, G_n converges to F pointwise. As X is compact we have that $G_n \rightarrow F$, uniformly. \blacksquare

Proposition 27. Let $T : X \rightarrow X$ be a continuous map such that $T : X \rightarrow T(X)$ is a homeomorphism. If $\lim_{n \rightarrow \infty} T^n(x) = p$, $\forall x \in X$, then $\lim_{n \rightarrow \infty} \Phi^n(\mu) = \delta_p$, $\forall \mu \in \mathcal{P}(X)$.

Proof. Take $\varepsilon > 0$. We need to show that there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow d(\Phi^n(\mu), \delta_{T^n(p)}) < \varepsilon.$$

By Lemma 26 we can see that given $\delta > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(T^n(x), p) < \delta$, for all $x \in X$ and $n > n_0$. Now we take $g \in C(X)$ and see that

$$\begin{aligned} \left| \int_X g(x) d(\Phi^n(\mu)) - \int_X g(x) d\delta_p \right| &= \left| \int_X (g(T^n(x)) - g(p)) d\mu \right| \\ &\leq \int_X |g(T^n(x)) - g(p)| d\mu \\ &\leq \sup_{x \in X} |g(T^n(x)) - g(p)|. \end{aligned}$$

Since $g \in C(X)$ we get $\left| \int_X g(x) d(\Phi^n(\mu)) - \int_X g(x) d\delta_p \right| \rightarrow 0$, for all $g \in C(X)$. Hence $d(\Phi^n(\mu), \delta_p) \rightarrow 0$. \blacksquare

6.2. Uniform attractor. In this section we define the concept of uniform attractor and see what happens with the dynamics Φ when T has a uniform attractor. To do that we suppose that X is separable.

Definition 28. Let $\Lambda \subseteq X$ be a compact set such that $T(\Lambda) \subseteq \Lambda$. We say that Λ is a uniform attractor for T , if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow d(T^n(x), \Lambda) < \varepsilon, \quad \forall x \in X.$$

Lemma 29. Let $T : X \rightarrow X$ be a homeomorphism from X to $T(X)$ and $A = \{a_j\}_{j \in \mathbb{N}}$ dense in X . Then $T^n(A)$ is dense in $T^n X$, for all $n \in \mathbb{N}$.

Proof. Take $x \in X$ and $\varepsilon > 0$, we have to show that for all $n \in \mathbb{N}$, there exists $a_i \in A$ such that $d(T^n(x), T^n(a_i)) < \varepsilon$. Since T^n is a continuous map, there exists $\delta > 0$ such that

$$d(y, a_i) < \delta \Rightarrow d(T^n(y), T^n(a_i)) < \varepsilon.$$

Using the density of A in X we get the result. ■

Lemma 30. (i) If $\mu = \sum_{i=1}^l \alpha_i \delta_{a_i}$, and $\nu = \sum_{i=1}^k \beta_i \delta_{b_i}$, then

$$d_P(\nu, \mu) \leq \max\{d(a_i, b_j)\}.$$

(ii) If $\mu = \sum_{i=1}^l \alpha_i \delta_{a_i}$ and $\nu = \sum_{i=1}^l \alpha_i \delta_{b_i}$, then

$$d_P(\nu, \mu) \leq \min\{d(a_i, b_i)\},$$

where d_P is the Prokhorov distance.

Proof. (i) We take $\gamma > \max\{d(a_i, b_j)\}$ and $A \in \mathcal{B}(X)$. Then

$$\exists a_i \in A \Rightarrow b_j \in A_\gamma, \quad \forall j, \quad \text{and} \quad \exists b_i \in A \Rightarrow a_j \in A_\gamma \quad \forall j.$$

Hence we have that

$$\mu(A) \leq \nu(A_\gamma) + \gamma, \quad \nu(A) \leq \mu(A_\gamma) + \gamma,$$

for all $A \in \mathcal{B}(X)$. Then, by the definition of d_P , we conclude

$$d_P(\nu, \mu) \leq \max\{d(a_i, b_j)\}.$$

(ii) We take $\gamma > \min\{d(a_i, b_i)\}$. We notice that

$$\exists a_i \in A \Rightarrow b_i \in A_\gamma, \quad \text{and} \quad \exists b_i \in A \Rightarrow a_i \in A_\gamma.$$

for all $A \in \mathcal{B}(X)$. Then, by the definition of d_P , we conclude

$$d_P(\nu, \mu) \leq \min\{d(a_i, b_i)\}.$$
■

Lemma 31. If X is a compact separable metric space then $\mathcal{P}(X)$ is a compact separable metric space.

Proof. Let A the enumerable dense set in X . Consider

$$\mathcal{A} = \left\{ \sum_{i=1}^k \alpha_i \delta_{x_i} : \alpha_1, \dots, \alpha_{i_l} \in [0, 1] \cap \mathbb{Q}, \quad x_i \in A \text{ and } k \in \mathbb{N} \right\}.$$

It is not difficult to see that \mathcal{A} is an enumerable dense set in $\mathcal{P}(X)$. ■

Theorem 32. *Let $\Lambda \subseteq X$ be an uniform attractor for T . If*

$$\mathcal{D} := \left\{ \sum_{i=1}^k \alpha_i \delta_{q_i} : \sum_{i=1}^k \alpha_i = 1, \quad q_i \in \Lambda, \quad \alpha_i \in [0, 1] \cap \mathbb{Q}, \quad k \in \mathbb{N} \right\},$$

then $\overline{\mathcal{D}}$ is an uniform attractor for Φ .

Proof. By Lemma 31, we have that

$$\mathcal{A} = \left\{ \sum_{i=1}^k \alpha_i \delta_{a_i} : \sum_{i=1}^k \alpha_i = 1, \quad a_i \in A, \quad \alpha_i \in [0, 1] \cap \mathbb{Q} \text{ and } k \in \mathbb{N} \right\}$$

is dense in $P(X)$. Using the Lemma 30

$$\lim_{n \rightarrow \infty} d(\Phi^n(\nu), \mathcal{D}) = 0,$$

uniformly, for all $\nu \in \mathcal{A}$. In fact, if we take $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow d(T^n(a_i), \Lambda) < \varepsilon, \quad \forall a_i \in A.$$

Given $a_i \in A$, there exists $q_i \in \Lambda$, such that $d(T^n(a_i), q_i) < \varepsilon$. Hence if $\nu = \sum_{i=1}^k \alpha_i \delta_{a_i}$ and we consider $\nu' = \sum_{i=1}^k \alpha_i \delta_{q_i}$, where $d(T^n(a_i), q_i) < \varepsilon$, we see that, by Lemma 30,

$$d_P(\Phi^n(\nu), \nu') < \min\{d(T^n(a_i), q_i)\} < \max\{d(T^n(a_i), q_i)\} < \varepsilon.$$

Now we take $\mu \in P(X)$ and $\varepsilon > 0$. We know that there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow d_P(\Phi^n(\nu), \mathcal{D}) < \varepsilon, \quad \forall \nu \in \mathcal{A},$$

then, using the continuity of Φ^n , we have that there exists $\delta > 0$ such that

$$d_P(\mu, \nu) < \delta \Rightarrow d_P(\Phi^n(\mu), \Phi^n(\nu)) < \varepsilon.$$

As \mathcal{A} is dense in X , there exists $\nu \in \mathcal{A}$, such that $d_P(\nu, \mu) < \delta$. Finally we get

$$n > n_0 \Rightarrow d_P(\Phi^n(\mu), \mathcal{D}) \leq d_P(\Phi^n(\nu), \mathcal{D}) + d_P(\Phi^n(\mu), \Phi^n(\nu)) < 2\varepsilon.$$

We observe that the last inequality is independent of $\mu \in P(X)$. ■

Example 33. *Consider $X = [0, 1] \times [0, 1]$ and $T : X \rightarrow X$ given by $T(x, y) = (x, (\frac{1}{2} + \frac{1}{2}x)y)$, then $\Lambda = \{(x, y) : x = 1, \text{ or } y = 0\}$ is a uniform attractor to T . In fact given $(x, y) \in X$,*

$$d(T^n(x, y), \Lambda) = d((x, \frac{(1+x)^n}{2^n}y), \Lambda) = \min\{1-x, \frac{(1+x)^n}{2^n}y\}.$$

If we take $0 < \varepsilon < 1$, we have that $x \leq \varepsilon$ or $\varepsilon < x$. If $\varepsilon < x$, then $1-x < \varepsilon$. If $x \leq \varepsilon$, then we can see that

$$\frac{(1+x)^n}{2^n} \leq \frac{(1+\varepsilon)^n}{2^n} \rightarrow 0.$$

It implies that there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow \frac{(1+x)^n}{2^n}y \leq \frac{(1+\varepsilon)^n}{2^n}y < \varepsilon.$$

Then we conclude that

$$n > n_0 \Rightarrow d(T^n(x, y), \Lambda) = \min\{1-x, \frac{(1+x)^n}{2^n}y\} < \varepsilon.$$

On the other hand, if we apply the Theorem 32, we get that the closure of

$$\mathcal{D} := \left\{ \sum_{i=1}^k \alpha_i \delta_{(x_i, y_i)} : \sum_{i=1}^k \alpha_i = 1, \right. \\ \left. (x_i, y_i) = (x_i, 0) \text{ or } (x_i, y_i) = (1, y_i), \quad \alpha_i \in [0, 1] \cap \mathbb{Q} \text{ and } k \in \mathbb{N} \right\},$$

is a uniform attractor to Φ .

Example 34. (Uniformly hyperbolic attractor) Consider the solid torus $\mathcal{T} = S^1 \times D^2$, where $S^1 = [0, 1] \bmod 1$ and $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. We fix $\lambda \in (0, \frac{1}{2})$ and define $T : \mathcal{T} \rightarrow \mathcal{T}$ by

$$T(\phi, x, y) = (2\phi, \lambda x + \frac{1}{2} \cos(2\pi\phi), \lambda y + \frac{1}{2} \sin(2\pi\phi)).$$

The map is injective and stretches by a factor of 2 in the S^1 -direction, contracts by a factor of λ in the D^2 -direction, and wraps the image twice inside \mathcal{T} .

The image $F(\mathcal{T})$ is contained in the interior $\text{int}(\mathcal{T})$ and $F^{n+1}(\mathcal{T}) \subset \text{int}(F^n(\mathcal{T}))$. A slice $F(\mathcal{T}) \cap \{\phi = c\}$ consists of two disks of radius λ centered at diametrically opposite points at distance $\frac{1}{2}$ from the center of the slice. A slice $F^n(\mathcal{T}) \cap \{\phi = c\}$ consists of 2^n -disks of radius λ^n : two disks inside each of 2^{n-1} disks of $F^{n-1}(\mathcal{T}) \cap \{\phi = c\}$.

The set $S = \bigcap_{n=0}^{\infty} F^n(\mathcal{T})$ is called a solenoid. It is a closed F -invariant subset of \mathcal{T} on which F is bijective. The solenoid is a uniform attractor for F . Moreover S is a hyperbolic set, then S is an example of an uniformly hyperbolic attractor.

Then, by Theorem 32, the closure of

$$\mathcal{D} := \left\{ \sum_{i=1}^k \alpha_i \delta_{q_i} : \sum_{i=1}^k \alpha_i = 1, \quad q_i \in S, \quad \alpha_i \in [0, 1] \cap \mathbb{Q} \text{ and } k \in \mathbb{N} \right\},$$

is a uniform attractor for Φ .

7. TOPOLOGICAL ENTROPY

Here we get a very interesting connection between the topological entropy of the map T and the topological entropy

We briefly recall the definition of the topological entropy.

Definition 35. Let $T : X \rightarrow X$ a continuous map. A subset $A \subset X$ is said (n, ε) -separated if any two distinct points x, y satisfy

$$d_n(x, y) := \max_{0 \leq k \leq n-1} d(T^k(x), T^k(y)) \geq \varepsilon.$$

Each d_n is a metric on X , moreover the d_i are all equivalent metrics.

Let us denote by $\text{sep}(T, n, \varepsilon)$ the maximal cardinality of a (n, ε) -separated set. Introducing

$$h_\varepsilon(T) = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \text{sep}(T, n, \varepsilon)$$

the topological entropy of the map T is then given by

$$h(T) = \lim_{\varepsilon \rightarrow 0} h_\varepsilon(T).$$

Now we can state some results about the topological entropy of the map Φ .

Lemma 36. *Let $T : X \rightarrow X$ be a continuous map such that $T : X \rightarrow T(X)$ is a homeomorphism. If $\lim_{n \rightarrow \infty} T^n(x) = p$, for all $x \in X$, then $h(T) = 0$.*

Proof. We know, by Lemma 26, if $\lim_{n \rightarrow \infty} T^n(x) = p$, then the sequence $\{T^n\}_{n \in \mathbb{N}}$ converges uniformly to the constant map $G \equiv p$. Let us take $A \subset X$ $(N_\varepsilon, \varepsilon)$ -separated with maximum cardinality and observe that A is (n, ε) -separated for all $n \geq N_\varepsilon$. Moreover if B is (n, ε) -separated, the cardinality of B is at most equals to the cardinality of A . Hence

$$h_\varepsilon(T) = \overline{\lim_{n \rightarrow \infty}} \frac{1}{n} \log \text{sep}(T, n, \varepsilon) = 0,$$

which implies

$$h(T) = \lim_{\varepsilon \rightarrow 0} h_\varepsilon(T) = 0. \quad \blacksquare$$

If we apply the Lemma 27 and after apply the Proposition 26 we can prove the following:

Theorem 37. *If $\lim_{n \rightarrow \infty} T^n(x) = p$, then $h(\Phi) = 0$, where $h(\Phi)$ is the topological entropy of Φ .*

Corollary 38. *If T is C -Lipschitz with $C < 1$, then $h(\Phi) = 0$.*

Proof. As T is C -Lipschitz, then Φ is C -Lipschitz, with $C < 1$. Then we have that $\lim_{n \rightarrow \infty} \Phi^n(\mu) = \delta_p$, where p is the fixed point for T , for all $\mu \in \mathcal{P}(X)$. Hence, by Lemma 26 $h(\Phi) = 0$. \blacksquare

Remark 39. *As we proved in Lemma 17, if T is C -Lipschitz, then Φ is C -Lipschitz. Hence if $C = 1$, then Φ is non-expansive and it implies that $h(\Phi) = 0$.*

Example 40. *Consider the map $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by $T(x) = x + \alpha$, α irrational, then $h(\Phi) = 0$. In fact, as T is an isometry we have that Φ is 1-Lipschitz, then $h(\Phi) = 0$.*

Definition 41. *The set of probability measures supported on a finite set is given by the union $\mathcal{D} = \cup_{n \geq 1} \mathcal{D}_n$, where*

$$\mathcal{D}_n = \left\{ \mu = \sum_{i=1}^n p_i \delta_{x_i} : (p_1, \dots, p_n) \in \mathbb{R}^n, \sum_{i=1}^n p_i = 1 \text{ and } x_i \in X \right\},$$

and for a fixed $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n p_i = 1$ we define the set

$$\mathcal{D}_n(p) = \left\{ \mu = \sum_{i=1}^n p_i \delta_{x_i} : x_i \in X \right\}.$$

We notice that is possible to make a copy of the space X in the $\mathcal{P}(X)$ as follows:

$$j : X \rightarrow \mathcal{D}_1 \subset \mathcal{P}(X)$$

$$x \mapsto \delta_x.$$

If we consider \mathcal{D}_1 , we notice that $\Phi(\mathcal{D}_1) = \mathcal{D}_1$, i.e., \mathcal{D}_1 is Φ -invariant.

Lemma 42. (See [7] and [4]) j is a homeomorphism onto \mathcal{D}_1 . If we consider the Wasserstein distance, j is an isometry.

Lemma 43. Let $S : Z \rightarrow Z$ a homeomorphism of a compact metric space. If $F \subset Z$ is a closed invariant subset of X then

$$h(S|_F) \leq h(S).$$

Proof. See [6]. ■

Proposition 44. $h(\Phi) \geq h(T)$.

Proof. We know that $j \circ T(x) = \delta_{T(x)} = \Phi \circ j(x)$, i.e.,

$$j \circ T = \Phi \circ j.$$

Hence T is topologically conjugated to Φ restricted to \mathcal{D}_1 , which implies

$$h(\Phi) \geq h(\Phi|_{\mathcal{D}_1}) = h(T),$$

because \mathcal{D}_1 is Φ -invariant. ■

We have another important relation between $h(T)$ and $h(\Phi)$. To prove this relation we need some results, which we will not prove.

Lemma 45. (Goodwin, 1971) Let X and Y compact Hausdorff spaces and let $T : X \rightarrow X$ and $S : Y \rightarrow Y$ continuous. Then

$$h(T \times S) = h(T) + h(S),$$

where h denotes the topological entropy and $T \times S : X \times Y \rightarrow X \times Y$ is defined as

$$(T \times S)(x, y) = (T(x), S(y)), \text{ for } (x, y) \in X \times Y.$$

Theorem 46. If $h(T) > 0$ then $h(\Phi) = \infty$.

Proof. Consider $n \in \mathbb{N}$ and $p \in \mathbb{R}^n$, such that $p = (p_1, \dots, p_n)$ and $p_i = \frac{2^{i-1}}{2^n - 1}$, and take the set $\mathcal{D}_n(p)$. We notice that $\mathcal{D}_n(p)$ is a closed subset of $\mathcal{P}(X)$, since $\mathcal{D}_n(p) = \sum_{i=1}^n p_i \mathcal{D}_1$. So we consider a map $\delta_p : X^n \rightarrow \mathcal{D}_n(p)$ defined as

$$\delta_p(x_1, \dots, x_n) := \sum_{i=1}^n p_i \delta_{x_i}.$$

We also consider the map $T^{(n)} : X^n \rightarrow X^n$ defined as

$$T^{(n)}(x_1, \dots, x_n) := (T(x_1), \dots, T(x_n))$$

It is not difficult to see that δ_p and $T^{(n)}$ are continuous, and they satisfy

$$\Phi \circ \delta_p = \delta_p \circ T^{(n)}.$$

We claim that δ_p is injective. In fact if $\delta_p(x) = \delta_p(y)$ and $y \neq x$, then

$$\left(\sum_{i=1}^n p_i \delta_{x_i} \right)(A) = \left(\sum_{i=1}^n p_i \delta_{y_i} \right)(A), \text{ for all open } A \subset X.$$

So there is k such that $x_k \neq y_k$. Take an open set A (we can do it because we are assuming X Hausdorff), such that $x_k \in A$ but $y_k \notin A$. We consider the set

of points $x_i \in A$, say $\{x_{i_1}, \dots, x_{i_l}\} \subset \{x_1, \dots, x_n\}$ that set. Using the same idea consider the set of points $y_j \in A$, say $\{y_{j_1}, \dots, y_{j_s}\} \subset \{y_1, \dots, y_n\}$ (observe that $y_k \notin \{y_{j_1}, \dots, y_{j_s}\}$). Then we have that

$$\sum_{t=1}^l \frac{2^{i_t-1}}{2^n-1} = \left(\sum_{i=1}^n p_i \delta_{x_i} \right)(A) = \left(\sum_{i=1}^n p_i \delta_{y_i} \right)(A) = \sum_{m=1}^s \frac{2^{j_m-1}}{2^n-1},$$

and it implies

$$\sum_{t=1}^l 2^{i_t-1} = \alpha = \sum_{m=1, j_m \neq k+1}^s 2^{j_m-1}.$$

Then we see that $\alpha \in \mathbb{N}$ has two different representations in base 2, it is a contradiction and we get δ_p injective. Clearly we have that δ_p is surjective, then δ_p is a bijection. As δ_p is continuous and X^n and $\mathcal{D}_n(p)$ are compact (because X is compact and $\mathcal{D}_n(p)$ is a closed subset of a compact set) we have that δ_p is a homeomorphism. As $\Phi \circ \delta_p = \delta_p \circ T^{(n)}$ and δ_p is a homeomorphism, δ_p is a conjugation. Then

$$nh(T) = h(T^{(n)}) = h(\Phi|_{\delta_p(X^n)}) \leq h(\Phi),$$

as $h(T) > 0$ we get the result. ■

Corollary 47. *If T is continuous and $h(T) > 0$ then $h(\Phi) = \infty$.*

Proof. We notice that we did not use the fact that T is a homeomorphism. So we got a homeomorphism $\delta_p : X^n \rightarrow \mathcal{D}_n(p)$ such that $\Phi \circ \delta_p = \delta_p \circ T^{(n)}$. It implies that $h(T^{(n)}) \geq h(\Phi|_{\mathcal{D}_n(p)})$. By the other hand we have that $\delta_p^{-1} \circ \Phi = T^{(n)} \delta_p^{-1}$. It implies that $h(T^{(n)}) \leq h(\Phi|_{\mathcal{D}_n(p)})$. Finally we get

$$nh(T) = h(T^{(n)}) = h(\Phi|_{\mathcal{D}_n(p)}) \leq h(\Phi).$$
■

Example 48. *Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and consider the map $\phi_d : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined by*

$$\phi_d(x) = dx \text{ mod } 1.$$

We know that $h(\phi) = \log d$. Then if Φ is the induced map by ϕ_d , so we have that $h(\Phi) = \infty$.

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